Imprints of Nuclear Symmetry Energy on Properties of Neutron Stars and Gravitational Waves

Bao-An Li

Collaborators:
Farrooh J. Fattoyev, Indiana University
Xiao-Tao He, Nanjing University of Aeronautics and Astronautics
Plamen G. Krastev, Harvard University
Che Ming Ko, Texas A&M University
Weikang Lin, Will Newton & Aaron Worley, TAMU-Commerce
Jun Xu, Chinese Academy of Sciences
De-Hua Wen, South China University of Technology
Naibo Zhang, Shandong University (Weihai)
Outline

What do we know about nuclear symmetry energy $E_{\text{sym}}(\rho)$?

What are the impacts of the $E_{\text{sym}}(\rho)$ on neutron stars & gravitational waves?
W.A. Zajc, arXiv:1707.01993

Probing the QCD phase diagram with relativistic heavy-ion collisions over the last 40 years

Fig. 1: A selection of representations of the QCD phase diagram in the ($\mu_B, T$) plane.
Empirical parabolic law of the EOS of cold, neutron-rich nucleonic matter

\[ E(\rho_n, \rho_p) = E_0(\rho_n = \rho_p) + E_{\text{sym}}(\rho) \left( \frac{\rho_n - \rho_p}{\rho} \right)^2 + o(\delta^4) \]

- Symmetry energy
- Isospin asymmetry \( \delta \)
- Energy per nucleon in symmetric matter
- Energy in asymmetric nucleonic matter

\[ \delta = \frac{\rho_n - \rho_p}{\rho} \]
Parameterizing the EOS of symmetric matter and symmetry energy for the NS core

\[ E_0(\rho) \approx E_0(\rho_0) + \frac{K_0}{2}(\frac{\rho - \rho_0}{3\rho_0})^2 + \frac{J_0}{6}(\frac{\rho - \rho_0}{3\rho_0})^3, \]

\[ E_{\text{sym}}(\rho) \approx E_{\text{sym}}(\rho_0) + L(\frac{\rho - \rho_0}{3\rho_0}) + \frac{K_{\text{sym}}}{2}(\frac{\rho - \rho_0}{3\rho_0})^2 + \frac{J_{\text{sym}}}{6}(\frac{\rho - \rho_0}{3\rho_0})^3 \]

Naturally approach asymptotically their Taylor expansions near the saturation density

**Current status of the restricted EOS parameter space:**

- **Low density:** \( K_0 = 240 \pm 20, \ E_{\text{sym}}(\rho_0) = 31.7 \pm 3.2 \) and \( L = 58.7 \pm 28.1 \) MeV
- **High density:** \(-400 \leq K_{\text{sym}} \leq 100, \ -200 \leq J_{\text{sym}} \leq 800, \) and \(-800 \leq J_0 \leq 400 \) MeV

**The minimum model of neutron stars: The pressure in the npeµ matter:**

\[ P(\rho, \delta) = P_0(\rho) + P_{\text{asy}}(\rho, \delta) = \rho^2 \left( \frac{\partial E}{\partial \rho} \right)_\delta + \frac{1}{4} \rho_e \mu_e \]

\[ = \rho^2 \left[ E'(\rho, \delta = 0) + E'_{\text{sym}}(\rho) \delta^2 \right] + \frac{1}{2} \delta(1 - \delta) \rho E_{\text{sym}}(\rho), \]

the \( \beta \)-equilibrium condition \( \mu_n - \mu_p = \mu_e = \mu_\mu \approx 4\delta E_{\text{sym}}(\rho) \)
What is the density dependence of symmetry energy?

Indications of experiments (model dependent)


Betty Tsang et al., PRC 86, 105803 (2012).
Chuck Horowitz et al., JPG: 41, 093001(2014)
Bao-An Li, Nuclear Physics News 27, 7 (2017)
Constraints on $E_{\text{sym}}(\rho_0)$ and L based on 29 analyses of data

**Fiducial values as of Aug. 2013**

$E_{\text{sym}}(\rho_0) \approx 31.6 \pm 2.66 \text{ MeV}$

$L \approx 2 E_{\text{sym}}(\rho_0) = 59 \pm 16 \text{ MeV}$

---

**Equation**

\[
E_{\text{sym}}(\rho) = E_{\text{sym}}(\rho_0) + \frac{L}{3} \left( \frac{\rho}{\rho_0} - 1 \right) + \frac{K_{\text{sym}}}{18} \left( \frac{\rho}{\rho_0} - 1 \right)^2 + \frac{J_{\text{sym}}}{162} \left( \frac{\rho}{\rho_0} - 1 \right)^3
\]

---

**References**

Constraints on $E_{\text{sym}}(\rho_0)$ and L based on 53 analyses of data

Fiducial values as of Oct. 12, 2016

$E_{\text{sym}}(\rho_0) \approx 31.7 \pm 3.2$ MeV

$L = 58.7 \pm 28.1$ MeV

Assuming:
(1) Gaussian distribution of L
(2) Democratic principle
(treat & trust everyone/publication equally)

M. Oertel, M. Hempel, T. Klähn, S. Typel
Review of Modern Physics 89 (2017) 015007
QMD analysis of GSI data on neutron-proton relative elliptical flow

\[ L = 85 \pm 22 \text{(exp)} \pm 20 \text{(th)} \pm 12 \text{(sys)} \text{ MeV} \]

\[ K_{\text{sym}} = 96 \pm 315 \text{(exp)} \pm 170 \text{(th)} \pm 166 \text{(sys)} \text{ MeV} \]

\[ E_{\text{sym}}(\rho) = E_{\text{sym}}(\rho_0) + \frac{L}{3}(\frac{\rho}{\rho_0} - 1) + \frac{K_{\text{sym}}}{18}(\frac{\rho}{\rho_0} - 1)^2 + \frac{J_{\text{sym}}}{162}(\frac{\rho}{\rho_0} - 1)^3 \]
Effects of symmetry energy on the crust-core transition density

Parameterized EOS for the core

NV+BPS EOS for the crust

At the crust-core transition: Incompressibility in neutron stars at $\beta$ equilibrium = 0

$$K_\mu = \rho^2 \frac{d^2 E_0}{d\rho^2} + 2\rho \frac{dE_0}{d\rho} + \delta^2 \left[ \rho^2 \frac{d^2 E_{sym}}{d\rho^2} + 2\rho \frac{dE_{sym}}{d\rho} - 2E_{sym}^{-1} \left( \rho \frac{dE_{sym}}{d\rho} \right)^2 \right]$$

Gravity-EOS Degeneracy in massive neutron stars

Strong-field gravity: GR or Modified Gravity?

Action $S = S_{\text{gravity}} + S_{\text{matter}}$

Matter+[Dark Matter]+[Dark Energy]

Contents and stiffness of the EOS of super-dense matter?

For high-density, cold neutron-rich nucleonic matter, the most uncertain part of the EOS is the nuclear symmetry energy
Effects of high-density symmetry energy on space-time curvatures (strength of gravity) of neutron stars

\[ \kappa^2 = \kappa^2 \left[ 3 \left( \mathcal{E}(r) + P(r) \right)^2 - 4 \mathcal{E}(r)P(r) \right] - \kappa \mathcal{E}(r) \frac{16GM(r)}{c^2r^3} + \frac{48G^2M^2(r)}{c^4r^6} \]

**What are Gravitational Waves?**

- Gravitational Waves = “Ripples in space-time”

Amplitude parameterized by (tiny) dimensionless strain $h$: $h(t) = DL/L$

The expected signal has the form (*P. Jaranowski, Phys. Rev. D58, 063001 (1998)*):

$$h(t) = F_+(t; \psi) h_0 \left(\frac{1 + \cos^2 \iota}{2}\right) \cos \Phi(t) - F_\times(t; \psi) h_0 \cos \iota \sin \Phi(t)$$

$F_+$ and $F_\times$: plus and cross polarization, bounded between -1 and 1

$h_0$ – amplitude of the gravitational wave signal, $\psi$ – polarization angle of signal

$\iota$ – inclination angle of source with respect to line of sight, $\Phi(t)$ - phase of pulsar

Proper separation between two masses
Possible sources of Gravitational Waves:

Examples

Compact binary inspiral: “chirps”

Elliptically deformed pulsars: “periodic”

Supernovae / GRBs: “bursts”

Non-radial oscillations of neutron stars
Effects of symmetry energy on the energy release during the hadron-QGP phase transition, frequency and damping time of the w-mode of gravitational waves in neutron stars

Weikang Lin, Bao-An Li, Jun Xu, Che Ming Ko, De-Hua Wen, PRC83, 045802 (2011)

Energy release: \( E_g = M_{g,h} - M_{g,q} \)

Strain amplitude: \( h(t) = h_0 e^{-\left(t/\tau_{gw}\right)} \sin(\omega_0 t) \),
Gravitational waves from elliptically deformed pulsars

Solving linearized Einstein’s field equation of General Relativity, the leading contribution to the GW is the mass quadrupole moment

$$h_0 = \chi \frac{\Phi_{22} \nu^2}{r}$$

Distance to the observer

Frequency of the pulsar

Breaking strain: fractional deformation when the crust fails

Mass quadrupole moment

$$\Phi_{22,\text{max}} = 2.4 \times 10^{38} \text{g cm}^2 \left( \frac{\sigma}{10^{-2}} \right) \left( \frac{R}{10\text{km}} \right)^{6.26} \left( \frac{1.4M_\odot}{M} \right)^{1.2}$$

Equatorial Ellipticity of pulsars

$$\epsilon = \sqrt{\frac{8\pi}{15}} \frac{\Phi_{22}}{I_{zz}}$$

$$\mathcal{E} = \frac{I_{xx} - I_{yy}}{I_{zz}}$$

S. Abbott et al., PRL 94, 181103 (05)
B.J. Owen, PRL 95, 211101 (05)
Nuclear constraints on the strength of gravitational waves


\[ h_0 \propto 10^{-24} \]

\[ \sigma = [10^{-5} - 10^{-2}] \quad \text{or} \quad 0.1 \]

\[ \Phi_{22,max} = 2.4 \times 10^{38} \text{ g cm}^2 \left( \frac{\sigma}{10^{-2}} \right) \left( \frac{R}{10\text{ km}} \right)^{6.26} \left( \frac{1.4M_{\odot}}{M} \right)^{1.2} \]
Estimate of gravitational waves from spinning-down of pulsars

Assumption: spinning-down is completely due to the GW radiation

\[ \frac{dE_{\text{rot}}}{dt} = 4\pi^2 I_{zz} \nu \dot{\nu} \]

assuming \( I_{zz} = 10^{38} \text{ kg m}^2 \) “Standard fiducial value”

\[ h_{\text{sd}} = \left( \frac{5}{2} \frac{GI_{zz} |\dot{\nu}|}{c^3 r^2 \nu} \right)^{1/2} \]

- Solid black lines: LIGO and GEO science requirement, for T=1 year
- Circles: upper limits on gravitational waves from known EM pulsars, obtained from measured spin-down
- Only known, isolated targets shown here

Testing the standard fudicial value of the moment of inertia assuming $I_{zz} = 10^{38}$ kg m$^2$

Aaron Worley, Plamen Krastev and Bao-An Li,
Signatures of high-density $E_{\text{sym}}$ in GW signals: Tidal deformability and mergers

- Tidal field $E_{ij}$ drives $f$-mode (quadrupole deformation $Q_{ij}$)
- Resulting energy transfer appears as phase shift in gravitational waveform
- Detectable cleanly $f\approx 100-400\text{Hz}$
- Phase shift depends on one parameter: tidal polarizability $\lambda$ (or Love number $k_2$)

Flanagan, Hinderer, PRD77, 021502 (2008)
Hinderer et al, PRD81, 123016 (2010)
Signatures of $S, L(n > n_0)$ in GW signals: Tidal deformability and mergers


$\Lambda$ of light NS is sensitive to $L$
$\Lambda$ of canonical and more massive NS is sensitive to high-density $E_{\text{sym}}$ but not $L$

- Detector sensitivities assuming optimally oriented, equal mass binary at $D=100$ Mpc
Imprints of nuclear $E_{\text{sym}}$ on the tidal deformability of neutron stars

Implications of GW170817

Plamen G. Krastev, Bao-An Li  

arXiv:1801.04620
Radii of canonical neutron stars from GW170817

Examples:

**arXiv:1805.11581** GW170817: Measurements of neutron star radii and equation of state, The LIGO Scientific Collaboration, the Virgo Collaboration: $R_2 = 11.9^{+1.4}_{-1.4}$ km at the 90% credible level.

**arXiv:1711.06615** Neutron skins and neutron stars in the multi-messenger era, F. J. Fattoyev, J. Piekarewicz, C. J. Horowitz
1.4 M⊙ neutron star of $R_{1.4} < 13.76$ km.

Elias R. Most, Lukas R. Weih, Luciano Rezzolla, Jürgen Schaffner-Bielich
1.4M⊙ is constrained to be $12.00 < R_{1.4}/\text{km} < 13.45$

**arXiv:1805.11963** GW170817: constraining the nuclear matter equation of state from the neutron star tidal deformability
Tuhin Malik, N. Alam, M. Fortin, C. Providência, B. K. Agrawal, T. K. Jha, Bharat Kumar, S. K. Patra; $11.82 \leq R_{1.4} \leq 13.72$ km.
Constraining the radii of neutron stars with terrestrial experiments

Radii of neutron stars inferred from observations
(a) thermal emissions from quiescent neutron star
low-mass X-ray binaries (qLMXBs)
(b) photospheric radius expansion (PRE) bursts
with H and/or He atmosphere models

<table>
<thead>
<tr>
<th>Model</th>
<th>$R_{1.4}$ 90% confidence range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alt/H+He QLMXB; $z = 0$ PRE</td>
<td>11.13 – 12.33</td>
</tr>
<tr>
<td>$z = 0$ PRE only</td>
<td>11.56 – 12.64</td>
</tr>
<tr>
<td>Base, QLMXB only</td>
<td>11.01 – 11.94</td>
</tr>
<tr>
<td>Alt, QLMXB only</td>
<td><strong>10.62 – 11.50</strong></td>
</tr>
<tr>
<td>H+He, QLMXB only</td>
<td>11.29 – 12.83</td>
</tr>
<tr>
<td>Alt/H+He, QLMXB only</td>
<td>11.24 – 12.59</td>
</tr>
</tbody>
</table>

J.M. Lattimer and A.W. Steiner,

Review of techniques & controversies:
M.C. Miller & F.K. Lamb,
European Physical Journal A 52, 63 (2016).

L.W. Chen, C.M. Ko and B.A. Li,
We have discovered a 716-Hz eclipsing binary radio pulsar in the globular cluster Terzan 5 using the Green Bank Telescope. It is the fastest-spinning neutron star ever found, breaking the 23-year-old record held by the 642-Hz pulsar B1937+21. The difficulty in detecting this pulsar, due to its very low flux density and high eclipse fraction (~40% of the orbit), suggests that even faster-spinning neutron stars exist. If the pulsar has a mass less than 2 M☉, then its radius is constrained by the spin rate to be <16 km. The short period of this pulsar also constrains models that suggest gravitational radiation, through an r-mode instability, limits the maximum spin frequency of neutron stars.

Recently, Li and Steiner (18) have derived a radius range of 11.5–13.6 km for a 1.4 M☉ neutron star, based on terrestrial laboratory measurements of nuclear matter. For a 1.4 M☉ neutron star, we find an upper limit of 14.4 km, which is in agreement with their result. These radius constraints are more robust than those obtained through observations of neutron star thermal emission, which is faint, difficult to measure, and whose characterization depends on uncertain atmosphere models (19). Although in principle a radius measurement could constrain the unknown equation of state of dense matter, PSR J1748–2446ad does not rule out any particular existing models, since the pulsar mass is unknown. It is unlikely that a
Assuming the observed frequency is the Kepler frequency

\[ f_k = \frac{1}{2\pi} \left( \frac{GM}{R_{eq}^3} \right)^{1/2} \]

Astrophysical constraints on the high-density EOS parameters in 3D

N.B. Zhang, B.A. Li and J. Xu, APJ 859, 90 (2018)

Degeneracy:
Intersections-> different EOS parameters
giving identical M&R, same R different M
or same M different R, have to fix $E_{\text{sym}}$ at HD to break it
Causality, not the blue sky, is the upper limit!
It is impossible to form a stable hyper-massive NS with $M=2.74M_{\odot}$
The maximum mass, radius and density on the causality surface

Isospin separation instability:

If $E_{\text{sym}} < 0$, replace $\delta=0$ with $1 + (-1)$

$\text{SNM} = \text{PNM} + \text{PPM}$ to lower energy

Tensor force in n-p due to $\rho$ meson exchange can make this happen

$$E(\rho, \delta) \approx E_0(\rho) + E_{\text{sym}}(\rho)\delta^2$$

Direct URCA limit

Canonical NSs have high core density, but almost pure neutrons $\rightarrow$ slow cooling
Summary

(1) Truly multi-messenger approach to probe the EOS of dense neutron-rich matter = astrophysical observations + terrestrial experiments + theories + …

(2) Symmetry energy has multiple imprints on properties of neutron stars and GWs
Can the symmetry energy become negative at high densities?

Yes, it happens when the tensor force due to $\rho$ exchange in the $T=0$ channel dominates. At high densities, the energy of pure neutron matter can be lower than symmetric matter leading to negative symmetry energy.

Example: proton fractions with interactions/models leading to negative symmetry energy


\[ x = 0.048 \left( \frac{E_{\text{sym}}(\rho)}{E_{\text{sym}}(\rho_0)} \right)^3 \left( \frac{\rho}{\rho_0} \right)(1 - 2x)^3 \]

Potential part of the symmetry energy

Tensor force and/or 3-body force can make $E_{\text{sym}}$ negative at high densities

3-body force effects in Gogny or Skyrme HF

\[ V_d = t_0(1 + x_0 P_\sigma)\rho^a \delta(r) \]

\[ E^{TBF}_{\text{sym}} = -(1 + 2x_0)^{1/8} \rho^{a+1} \]
The proton fraction \( x \) at \( \beta \)-equilibrium in proto-neutron stars is determined by

\[
x = 0.048 \left[ \frac{E_{\text{sym}}(\rho)}{E_{\text{sym}}(\rho_0)} \right]^3 (\rho / \rho_0) (1 - 2x)^3
\]

The critical proton fraction for direct URCA process to happen is \( X_p = 0.14 \) for npe\( \mu \) matter obtained from energy-momentum conservation on the proton Fermi surface.

Slow cooling: modified URCA:

\[
n + (n, p) \rightarrow p + (n, p) + e^- + \bar{\nu}_e
\]
\[
p + (n, p) \rightarrow n + (n, p) + e^+ + \nu_e
\]

Consequence: long surface thermal emission up to a few million years

Faster cooling by 4 to 5 orders of magnitude: direct URCA

\[
n \rightarrow p + e^- + \bar{\nu}_e
\]
\[
p \rightarrow n + e^+ + \nu_e
\]

The graph shows the relationship between the normalized energy density derivative $dP/d\varepsilon$ and the density $\rho$ in $[\text{fm}^{-3}]$. Different curves represent different values of $K_{\text{sym}}$: $K_{\text{sym}} = 100$ (black), $K_{\text{sym}} = 0$ (red), $K_{\text{sym}} = -100$ (blue), $K_{\text{sym}} = -200$ (orange), $K_{\text{sym}} = -300$ (green), and $K_{\text{sym}} = -400$ (olive). Additionally, the parameters $J_0 = 0$ and $J_{\text{sym}} = 800$ are indicated at the bottom right corner of the graph.
Among the promising observables of high-density symmetry energy:

- $\pi^-/\pi^+$, neutron-proton differential flow in heavy-ion collisions
- Radii of neutron stars
- Neutrino flux of supernova explosions
- Strain amplitude and frequency of gravitational waves from spiraling neutron star binaries and/or oscillations/rotations of deformed pulsars
CUSTIPEN—Xiamen Workshop on the EOS of Dense Neutron-Rich Matter in the Era of Gravitational Wave Astronomy, Jan. 3–7, 2019, Xiamen, China

Organized Committee:
Pawel Danielewicz (Michigan State University)
Ang Li (Co-Chair, Xiamen University)
Bao-An Li (Co-Chair, Texas A&M University-Commerce)
Taotao Fang (Xiamen University)
Jorge Piekarewicz (Florida State University)
Furong Xu (Peking University)
Renxin Xu (Peking University)
Bing Zhang (University of Nevada)
**Challenge**: how can neutron stars be stable with a super-soft symmetry energy?

If the symmetry energy is too soft, then a mechanical instability will occur when $\frac{dP}{d\rho}$ is negative, neutron stars will then all collapse while they do exist in nature.

For npe matter

\[
P(\rho, \delta) = P_0(\rho) + P_{asym}(\rho, \delta) = \rho^2 \left( \frac{\partial E}{\partial \rho} \right)_\delta + \frac{1}{4} \rho e \mu_e
\]

\[
= \rho^2 \left[ E'(\rho, \delta = 0) + E'_{sym}(\rho) \delta^2 \right] + \frac{1}{2} \delta(1 - \delta) \rho E_{sym}(\rho)
\]

$dP/d\rho < 0$ if $E'_{sym}$ is big and negative (super-soft) $\rightarrow$ Mechanical Instability

---

GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral

B. P. Abbott et al.*

(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 26 September 2017; revised manuscript received 2 October 2017; published 16 October 2017)
Supersoft Symmetry Energy Encountering Non-Newtonian Gravity in Neutron Stars

\[ g^2 / \mu^2 = 50 \text{ GeV}^{-2} \] to support
a NS of 1.4Msun and \( R = 12 \text{ km} \)
Why we **DO NOT** use piecewise polytropes at high densities as most others do?

1. The ranges of HD parameters $K_{\text{sym}}$, $J_{\text{sym}}$ and $J_0$ are so large that they cover ........
2. The polytropes: pressures at several fiducial densities have NO isospin dependence
3. Need a parameterization facilitating the extraction of high density symmetry energy

Piecewise polytropes: in each region

$$p = K \rho^{1+1/n}$$
Isoscalar Excitation Modes of Nuclear Resonance

$K_0 = 240 \pm 20$ MeV

\[
E_{\text{ISGMR}} \approx \sqrt{\frac{K_A}{m < r^2 >}}
\]

\[
E_{\text{ISGDR}} \approx \sqrt{\frac{3 K_A + (27/25) \varepsilon_F}{7 m < r^2 >}}
\]

\[
K_A = K_{\text{vol}} + K_{\text{surf}} A^{-1/3} + K_\tau \delta^2 + K_{\text{Coul}} \frac{Z^2}{A^{4/3}}
\]

Isospin dependence of incompressibility

\[
K_\tau = K_{\text{sym}} - 6L(\rho_0) - \frac{J_0 L(\rho_0)}{K_0}
\]

$K_\tau = -550 \pm 100$ MeV

Nothing conclusive about $K_{\text{sym}}$

An example of **EOS-Gravity degeneracy**


- Neutron stars are among the densest objects with the strongest gravity
- General Relativity (GR) may break down at strong-field limit and there is no fundamental reason to choose Einstein’s GR over alternative gravity theories
- Need at least 2 observables to break the degeneracy

Systematics from over 520 Skyrme+RMF energy density functionals

Ingo Tews, James M. Lattimer, Akira Ohnishi, Evgeni E. Kolomeitsev
Astrophysics J. 848, 105 (2017)

\[ E_{\text{sym}}(\rho) = E_{\text{sym}}(\rho_0) + \frac{L}{3} \left( \frac{\rho}{\rho_0} - 1 \right) + \frac{K_{\text{sym}}}{18} \left( \frac{\rho}{\rho_0} - 1 \right)^2 + \frac{J_{\text{sym}}}{162} \left( \frac{\rho}{\rho_0} - 1 \right)^3 \]
Setting $J_0=0$, $J_{\text{sym}}=0$, parameterizing the EOS quadratically in $\rho$ and $\delta$ in 2D.

Skewness $J_0$ of symmetric matter

Indications of model analyses of data

Current status of the restricted EOS parameter space:

Low density: $K_0 = 240 \pm 20$, $E_{\text{sym}}(\rho_0) = 31.7 \pm 3.2$ and $L = 58.7 \pm 28.1$ MeV

High density: $-400 \leq K_{\text{sym}} \leq 100$, $-200 \leq J_{\text{sym}} \leq 800$, and $-800 \leq J_0 \leq 400$ MeV


Cai et al.

Skewness $J_0$

Supersoft Symmetry Energy Encountering Non-Newtonian Gravity in Neutron Stars

$g^2 / \mu^2 = 50 \text{ GeV}^{-2}$ to support a NS of $1.4\text{M}_\odot$ and $R = 12 \text{ km}$